JOURNAL OF DIFFERENTIAL EQUATIONS 30, 1–19 (1978)

Nonlinear Perturbations of a Linear Elliptic Problem Near Its First Eigenvalue

DJAIRO G. DE FIGUEIREDO

Departamento de Matematica, Universidade de Brasilia, Instituto de Ciencias Exatas, 70,000 Brasilia, D. Federal, Brazil

AND

JEAN-PIERRE GOSSEZ

Department of Mathematics, University of Brussels, Brussels, Belgium

Received January 20, 1977; revised November 1, 1977

1. INTRODUCTION

In this paper we investigate the existence of solutions for the Dirichlet problem

$$Lu = f(x, u), \qquad \text{in } \Omega \tag{1}$$

$$\partial^j u/\partial \nu^j = 0 \qquad (j = 0, 1,..., m - 1), \qquad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\partial/\partial\nu$ is the derivative in the direction of the outward normal to the boundary $\partial\Omega$ of Ω, and L is a symmetric uniformly strongly elliptic operator of order $2m$, $m \geqslant 1$. The main point here is a successful treatment of cases when there is a fast growth of f as $u \to \pm\infty$, as well as a certain interaction of f with the spectrum of L, giving rise to resonance. In two previous papers, [1, 2], the first named author has considered problems of the type studied here, but he has essentially been able to treat the case when f grows linearly as $u \to \pm\infty$. When $m = 1$, results similar to the ones presented here have been obtained by Kazdan and Warner [3], using completely different techniques. We emphasize that the method used in [3] cannot be extended to treat higher order equations, since it is based on maximum principles only available for second-order elliptic equations. We establish the existence of generalized solutions only; very little is known about regularity for higher order equations, cf. [16].

A very crucial role will be played here by the limits $k_-(x)$ and $k_+(x)$ defined by

$$k_\pm(x) = \limsup_{s\to\pm\infty} \frac{f(x, s)}{s}, \tag{3}$$

0022-0396/78/0301–0001$02.00/0

which, in the situations considered below, will take values in $\mathbb{R} \cup \{-\infty\}$. If both k_- and k_+ are strictly less than the first eigenvalue λ_1 of L we have nonresonant problems, and the novelty seen in the present work is the possibility of treating cases when either or both k_- and k_+ are equal to $-\infty$ on sets of positive measure. We remark that, since f is not assumed to be decreasing, a direct use of the monotone operator theory in the sense of Browder–Minty is precluded. See Theorem 1 below. We remark also that the results of the second named author [4, 5] on elliptic problems in Orlicz–Sobolev spaces do not apply here; indeed the main point in those results was the introduction of rapidly (or slowly) increasing nonlinearities in the upper order terms.

If either k_- or k_+ equals λ_1 on some set of positive measure, resonance arises. In Section 4 we treat such problems even when the other k_- or k_+ equals $-\infty$ on sets of positive measure. Consequently our Theorem 4 extends the results of [2]. An anouncement of the results presented here appeared in [6]. After the completion of this paper we have been informed that a result related to our Theorem 4 has been obtained by Brezis–Nirenberg [16] using different techniques.

2. Some Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$, and

$$Lu = \sum_{|\alpha|,|\beta|\leqslant m} (-1)^{|\beta|} D^\beta(a_{\alpha\beta}(x)\, D^\alpha u)$$

be a differential operator acting on real-valued functions $u(x)$ defined in Ω, satisfying the following conditions.

(L-1) L is uniformly elliptic, that is, there is a constant $c_0 > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\, \xi^\alpha \xi^\beta \geqslant c_0 \mid \xi \mid^{2m}$$

for all $x \in \Omega$, and all $\xi \in \mathbb{R}^N$ ($\mid \xi \mid^2 = \xi_1{}^2 + \cdots + \xi_N{}^2$).

(L-2) The coefficients $a_{\alpha\beta}$ are real-valued functions in $L^\infty(\Omega)$ and the higher order coefficients $a_{\alpha\beta}$, $\mid \alpha \mid = \mid \beta \mid = m$, are uniformly continuous functions in Ω.

(L-3) L is symmetric, that is, $(L\phi, \psi)_0 = (\phi, L\psi)_0$, for all $\phi, \psi \in C_0{}^\infty(\Omega)$. $(\ ,\ )_0$ denotes L^2-inner product. Under assumptions (L-1) and (L-2) we have Gårding's inequality: there are real constants $c > 0$ and c' such that:

$$a[\phi, \phi] \geqslant c \,\|\phi\|_m^2 - c' \,\|\phi\|_0^2, \qquad \phi \in C_0{}^\infty(\Omega),$$

where $a[\cdot, \cdot]$ is the so-called Dirichlet bilinear form associated to L

$$a[\phi, \psi] = \int_\Omega \sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\alpha\phi(x)\, D^\beta\psi(x)\, dx,$$

and $\|\cdot\|_m$ is the Sobolev norm

$$\|\phi\|_m^2 = \int_\Omega \sum_{|\alpha| \leqslant m} |D^\alpha\phi(x)|^2\, dx.$$

Let us denote by $H_0^m(\Omega)$, or simply H_0^m, the completion of $C_0^\infty(\Omega)$ with respect to the $\|\cdot\|_m$-norm. In particular, $H_0^0(\Omega) = L^2(\Omega)$. It is easy to see that H_0^m is a Hilbert space under the following inner product

$$(u, v)_m = \int_\Omega \sum_{|\alpha| \leqslant m} D^\alpha u(x)\, D^\alpha v(x)\, dx.$$

The generalized Dirichlet problem for L is defined as follows. "Given $g \in L^2(\Omega)$, find $u \in H_0^m(\Omega)$ such that $a[u, \phi] = (g, \phi)_0$, for all $\phi \in C_0^\infty(\Omega)$." We see that such a u is a solution of $Lu = g$ in the distribution sense.

A real number λ is an eigenvalue for L if there is a nonzero distribution solution, $u \in H_0^m$, of $Lu = \lambda u$. From now on we also assume (L-3). As a consequence of Gårding's inequality, Lax–Milgran lemma, the Riesz–Schauder theory of linear compact operators, and the theory of symmetric compact operators in Hilbert spaces, we have the following conclusions. L has a sequence of (real) eigenvalues $\lambda_1 < \lambda_2 < \cdots$, with $\lambda_n \to \pm\infty$, whose corresponding eigenspaces are finite-dimensional subspaces of H_0^m. For each $g \in L^2$, the equation $Lu = g$ has one and only one distribution solution $u \in H_0^m$, provided 0 is not an eigenvalue of L. Moreover we have the following inequality

$$a[u, u] \geqslant \lambda_1 \|u\|_0^2, \qquad u \in H_0^m. \tag{4}$$

Details and proofs of most of the forementioned results can be seen in Friedman [7].

The nonlinearity is a function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ which is always assumed to satisfy

(f-1) Carathéodory's conditions, that is, for each fixed $s \in \mathbb{R}$, the function $x \to f(x, s)$ is measurable, and for almost all $x \in \Omega$ fixed, the function $s \to f(x, s)$ is continuous.

(f-2) For each $r > 0$, there is a function $\alpha_r \in L^2(\Omega)$ such that $|f(x, s)| \leqslant \alpha_r(x)$, for all $x \in \Omega$ and all $|s| \leqslant r$.

The point of assumption (f-1) is to obtain $f(x, u(x))$ as a measurable function of x, provided u is. Assumption (f-2) is to prevent certain wild behaviors of f

with respect to x. For example, we do not want functions like $f(x, s) = sx^{-1}$, when $\Omega = (0, 1)$. Observe that (f-2) is not a restriction whatsoever to the growth of f as $|s| \to \infty$; it is trivially satisfied when f does not depend on x: $f(x, s) = f(s)$.

As we remarked in the Introduction an important rôle is played by the positioning of the two limits defined in (3) with respect to the eigenvalues of L.

Nonresonant problems are those where k_- and k_+ are away of the eigenvalues. In [1] it is considered the case when

$$|f(x, s) - \lambda s| \leqslant \gamma |s| + b(x) \tag{$*$}$$

where $b(x) \in L^2(\Omega)$, $\lambda_n < \lambda < \lambda_{n+1}$, $0 < \gamma < \min(\lambda - \lambda_n, \lambda_{n+1} - \lambda)$, and λ_n, λ_{n+1} are two consecutive eigenvalues of L. The above inequality implies, the following two inequalities:

$$\lambda_n < \eta_n \leqslant k_-, \; k_+ \leqslant \eta_{n+1} < \lambda_{n+1}, \tag{$**$}$$

or

$$\lambda_n < \eta_n \leqslant l_-, \; l_+ \leqslant \eta_{n+1} < \lambda_{n+1} \tag{$***$}$$

where $l_\pm$ is defined as $k_\pm$, with lim sup replaced by lim inf. In this paper the nonresonant problem corresponds to the following assumption:

(f-3) $\qquad k_-, k_+ \leqslant \eta_1 < \lambda_1, \qquad$ uniformly in x,

where λ_1 is the first eigenvalue of L. This assumption does not imply that f has a linear growth as $s \to \pm\infty$. By (f-3) we mean more precisely that there is an η, with $\eta_1 < \eta < \lambda_1$, an $s_0 > 0$ and $0 \leqslant \beta(x) \in L^2(\Omega)$ such that

$$\frac{f(x, s)}{s} \leqslant \eta + \frac{\beta(x)}{|s|}, \qquad x \in \Omega, \; |s| \geqslant s_0, \tag{5}$$

which is equivalent to the two relations below

$$f(x, s) \leqslant \eta s + \beta(x), \qquad x \in \Omega, \quad s \geqslant s_0, \tag{6}$$

and

$$f(x, s) \geqslant \eta s - \beta(x), \qquad x \in \Omega, \quad s \leqslant -s_0. \tag{7}$$

So the growth of f as $s \to +\infty$ is bounded above by a linear function, but it is unrestricted from below. A similar statement can be made for the growth of f as $s \to -\infty$.

In the next section we shall prove the following theorem.

THEOREM 1. *Assume* (L-1), (L-2), (L-3), (f-1), (f-2), *and* (f-3). *Then there exists* $u \in H_0^m$, *with* $f(x, u(x)) \in L^1(\Omega)$ *and* $f(x, u(x))\, u(x) \in L^1(\Omega)$ *which is a distribution solution of* $Lu = f(x, u)$.

Remark. Assumptions (f-2) and (f-3) do not imply that the Niemytskii operator $F: u \to f(x, u)$ maps $L^2(\Omega)$ into itself. This causes a problem when, following Dolph [8] and [1], one tries to use the theory of (nonlinear) compact operators to solve a Hammerstein equation of the form $u = AFu$, which is equivalent to the Dirichlet problem for $Lu = f(x, u)$, with A essentially equal to L^{-1}. To overcome this difficulty we will approximate F by Niemytskii maps which are well defined in L^2.

Resonance arises when either k_+, k_-, l_+, or l_- are equal to eigenvalues on sets of positive measure, or when a crossing of eigenvalues occurs, such as $k_- < \lambda_n < k_+$. These problems are much more difficult, and they are not yet completely understood. The case $l_- = k_- = l_+ = k_+ = \lambda_n$ has been treated by several authors, [9, 10, 11, 1], and others. Again this implies a linear growth of f as $s \to \pm\infty$. For a recent survey on problems of this sort we mention Fucik [12]. The crossing of the first eigenvalue has been studied by Ambrosetti and Prodi [13]. In the present paper the resonant problem considered corresponds to the assumption

$$\text{(f-4)} \qquad k_-, k_+ \leqslant \lambda_1, \qquad \text{uniformly in } x,$$

with either k_- or k_+, or both, being equal to λ_1 on sets of positive measure. The possibility of also having k_+ or k_- equal to $-\infty$ somewhere is not excluded.

3. Proof of Theorem 1

Observe initially that we can without loss of generality assume that $\lambda_1 > 0$. Indeed, equation $Lu = f(x, u)$ is equivalent to $L_1 u = f_1(x, u)$, where $L_1 u = Lu + \lambda u$ and $f_1(x, u) = f(x, u) + \lambda u$. If λ is sufficiently large the first eigenvalue $\lambda_1 + \lambda$ of L_1 is positive, and the function f_1 satisfies the hypotheses (f-1), (f-2), and (f-3) with respect to L_1. The number η_1 of condition (f-3) will accordingly also be assumed >0.

We shall use the following lemma which is a special case of Theorem 1.

Lemma 1. *Assume* (L-1), (L-2), (L-3), (f-1), *and* (f-3). *Moreover suppose that f has a linear growth, that is, there is a constant $a > 0$ and an $L^2(\Omega)$-function $b(x)$ such that $|f(x, s)| \leqslant a|s| + b(x)$, for all $x \in \Omega$ and all $s \in \mathbb{R}$. Then there is a distribution solution $u \in H_0^m$ of $Lu = f(x, u)$.*

This lemma has been proved in [2] when $\beta(x) \equiv 0$ using Leray–Schauder fixed point theorem; the proof given there carries over immediately to the present situation. To prove the general case we approximate f by functions f_n which have a linear growth. For each positive integer n define

$$f_n(x, s) = \begin{cases} f(x, s), & \text{if } |s| \leqslant n \\ f(x, n), & \text{if } s \geqslant n \\ f(x, -n), & \text{if } s \leqslant -n. \end{cases} \tag{8}$$

It follows readily that

f_n satisfies Carathéodory's conditions, (8i)

$f_n(x, s)/s \leqslant \eta + \beta(x)/|s|$, for all $x \in \Omega$, and all $|s| \geqslant s_0$, (8ii)

$f_n(x, s)$ has a linear growth, or even better it is bounded by an L^2-function. For, in view of (f-2), one has $|f_n(x, s)| \leqslant \alpha_n(s)$, for all $x \in \Omega$ and all $s \in \mathbb{R}$. (8iii)

So Lemma 1 implies the existence of a distribution solution $u_n \in H_0^m$ of $Lu_n = f_n(x, u_n)$. It is also obtained that $f_n(x, u_n(x)) \in L^2(\Omega)$.

LEMMA 2. *The following estimates hold true*

$$\| u_n \|_0 \leqslant K, \tag{9}$$

$$0 \leqslant \int_\Omega f_n(x, u_n)u_n \leqslant K', \tag{10}$$

$$\| u_n \|_m \leqslant K'', \tag{11}$$

where K, K', and K'' are constants independent of n.

Proof. Using (4) we have

$$\lambda_1 \| u_n \|_0^2 \leqslant a[u_n , u_n] = \int_\Omega f_n(x, u_n)u_n \tag{12}$$

and using (5) and (f-2)

$$\int_{|u_n|<s_0} f_n(x, u_n)u_n + \int_{|u_n|\geqslant s_0} f_n(x, u_n)u_n$$
$$\leqslant s_0 \| \alpha_{s_0} \|_0 \, | \Omega |^{1/2} + \eta \| u_n \|_0^2 + \| \beta \|_0 \| u_n \|_0 , \tag{13}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Since $\lambda_1 > \eta$, inequalities (12) and (13) readily imply the estimates (9) and (10). Finally using Gårding's inequality and estimates (9) and (10) we obtain (11). Q.E.D.

Estimate (11) allows us to assert, passing to subsequences if necessary, that there exists $u \in H_0^m$ such that $u_n \to u$, weakly in H_0^m, that $u_n \to u$, in L^2 and that $u_n \to u$, a.e. in Ω. Thus $a[u_n , \phi] \to a[u, \phi]$ for all $\phi \in C_0^\infty(\Omega)$. So if we can prove that $f(x, u(x)) \in L^1(\Omega)$ and that $f_n(x, u_n(x)) \to f(x, u(x))$ in L^1, the existence of a distribution solution of $Lu = f(x, u)$ will be established. For that matter we first observe that $f_n(x, u_n(x)) \to f(x, u(x))$, a.e. in Ω. Indeed, let $x \in \Omega$, be fixed, and take n_0 such that $|u_n(x)| < |u(x)| + 1$ for all $n \geqslant n_0$; then for $n \geqslant \max\{n_0 , |u(x)| + 1\}$ we have $f_n(x, u_n(x)) = f(x, u(x))$, in view of (8). Thus we have just to prove Lemma 3 below and then the result follows from Vitali's theorem, cf. Dunford–Schwartz [14, p. 150]. The proof of Lemma 3 uses an idea of Strauss [15] and has been simplified after a suggestion of A. Damlamian.

LEMMA 3. *The functions $g_n(x) = f_n(x, u_n(x))$ are equiabsolutely integrable, that is, given $\epsilon > 0$ there is $\delta > 0$ such that for every $\Omega_0 \subset \Omega$, with Lebesgue measure $|\Omega_0| < \delta$, one has $\int_{\Omega_0} |g_n| < \epsilon$, for all n.*

Before proving the above lemma, let us introduce a partition of Ω made up of the following sets

$$\begin{aligned} A_n &= \{x \in \Omega \colon |u_n(x)| < s_0\} \\ B_n &= \{x \in \Omega \colon |u_n(x)| \geqslant s_0 \,;\, u_n(x)\, g_n(x) \geqslant 0\} \\ C_n &= \{x \in \Omega \colon |u_n(x)| \geqslant s_0 \,;\, u_n(x)\, g_n(x) < 0\}. \end{aligned}$$

and let us establish the estimates

$$\left| \int_{A_n} g_n u_n \right| \leqslant K_1 , \tag{14}$$

$$0 \leqslant \int_{B_n} g_n u_n \leqslant K_2 , \tag{15}$$

$$0 \leqslant \int_{C_n} -g_n u_n \leqslant K_3 , \tag{16}$$

where K_1, K_2, and K_3 are constants independent of n. Estimate (14) is a direct consequence of (f-2), and, in fact, K_1 can be taken as $s_0 \| \alpha_{s_0} \|_0 \, |\Omega|^{1/2}$, cf. (13). Estimate (15) follows directly from (8ii), and, in fact, K_2 can be taken as $\eta K^2 + K \| \beta \|_0$, where K is the constant in (9). Finally, (16) follows from (10), (14), and (15), and K_3 can be taken as $\eta K^2 + s_0 \| \alpha_{s_0} \|_0 \, |\Omega|^{1/2} + K \| \beta \|_0 + K'$, where K' is the constant in (10).

Proof of Lemma 3. For the given $\epsilon > 0$, choose $r \geqslant s_0$ such that

$$(1/r)[\eta K^2 + s_0 \| \alpha_{s_0} \|_0 \, |\Omega|^{1/2} + K \| \beta \|_0 + K'] < \epsilon,$$

and let

$$\begin{aligned} A_n' &= \{x \in \Omega : |u_n(x)| < r\} \\ B_n' &= \{x \in \Omega : |u_n(x)| \geqslant r, \quad u_n(x)\, g_n(x) \geqslant 0\} \\ C_n' &= \{x \in \Omega : |u_n(x)| \geqslant r, \quad u_n(x)\, g_n(x) < 0\}. \end{aligned}$$

We break Ω_0 as the union $(\Omega_0 \cap A_n') \cup (\Omega_0 \cap B_n') \cup (\Omega_0 \cap C_n')$, and estimate the integral of $|g_n|$ over each one of these sets; it suffices to consider $n \geqslant r$. First, from (f-2) and (8) we have:

$$\int_{\Omega_0 \cap A_n'} |g_n| = \int_{\Omega_0 \cap A_n'} |f(x, u_n)| \leqslant \| \alpha_r \|_{L^1(\Omega_0)} \,.$$

Next, using (15)

$$\int_{\Omega_0 \cap B_n'} | g_n | \leqslant \frac{1}{r} \int_{\Omega_0 \cap B_n'} | u_n | \, | g_n | = \frac{1}{r} \int_{\Omega_0 \cap B_n'} u_n g_n \leqslant \frac{1}{r} \int_{B_n} u_n g_n$$
$$\leqslant \frac{1}{r} K_2 < \epsilon,$$

and finally using (16)

$$\int_{\Omega_0 \cap C_n'} | g_n | \leqslant \frac{1}{r} \int_{\Omega_0 \cap C_n'} | u_n | \, | g_n | = \frac{1}{r} \int_{\Omega_0 \cap C_n'} -u_n g_n \leqslant \frac{1}{r} \int_{C_n} -u_n g_n$$
$$\leqslant \frac{1}{r} K_3 < \epsilon.$$

Now the proof will be complete by choosing $\delta > 0$ in such a way that $\| \alpha_r \|_{L^1(\Omega_0)} < \epsilon$ and $\int_{\Omega_0} | g_n | < \epsilon$, $n < r$, for all $| \Omega_0 | < \delta$. Q.E.D.

The proof of Theorem 1 will be completed by proving the final lemma.

LEMMA 4. *The function $f(x, u(x))\, u(x)$ is in $L^1(\Omega)$.*

Proof. Let us denote by h^+ and h^-, respectively, the positive and negative parts of a function $h(x)$, $x \in \Omega$. Since $f(x, u_n)u_n \to f(x, u)u$, a.e. in Ω, it follows that $[f(x, u_n)u_n]^+ \to [f(x, u)u]^+$, a.e. in Ω, and similarly for the negative parts. Now

$$\int_\Omega [f_n(x, u_n)u_n]^+ \leqslant \int_{A_n} | f_n(x, u_n)u_n | + \int_{B_n} f_n(x, u_n)u_n ,$$

which is uniformly bounded in view of estimates (14) and (15). Thus, by Fatou's lemma

$$\int_\Omega [f(x, u)u]^+ < \infty.$$

On the other hand

$$\int_\Omega [f_n(x, u_n)u_n]^- \leqslant \int_{A_n} | f_n(x, u_n)u_n | + \int_{C_n} -f_n(x, u_n)u_n ,$$

which is again uniformly bounded in view of estimates (14) and (16). Using Fatou's lemma once more we get

$$\int_\Omega [f(x, u)u]^- < \infty,$$

and Lemma 4 is proved. Q.E.D.

4. Problems wiht Resonance

As already mentioned, resonance in this paper corresponds to the following situation

$$\limsup_{|s|\to\infty} \frac{f(x, s)}{s} \leqslant \lambda_1 \text{, uniformly in } x, \tag{f-4}$$

with equality holding on subsets of Ω with positive measure. By (f-4) we mean more precisely that, given $\epsilon > 0$, there is $s_0 > 0$ and $0 \leqslant \beta(x) \in L^2(\Omega)$ such that

$$\frac{f(x, s)}{s} \leqslant \lambda_1 + \epsilon + \frac{\beta(x)}{|s|}, \; x \in \Omega, \; |s| \geqslant s_0 . \tag{17}$$

The idea for treating the Dirichlet problem (1)–(2) for functions satisfying (17) is to approximate Eq. (1) by nonresonant equations of the type $Lu + (1/n)u = f(x, u)$, whereupon Theorem 1 applies. This is a kind of argument originally used by Hess [11]. We start with the following auxiliary result.

Theorem 2. *Assume* (L-1), (L-2), (L-3), (f-1), (f-2), *and* (f-4). *Then the approximating equations*

$$Lu_n + \frac{1}{n} u_n = f(x, u_n) \tag{18}$$

have distribution solutions $u_n \in H_0^m$, *with* $f(x, u_n) \in L^1$ *and* $f(x, u_n)u_n \in L^1$. *Moreover, either*

(i) $\| u_n \|_m$ *is bounded and then* $Lu = f(x, u)$ *has a distribution solution* $u \in H_0^m$ *with* $f(x, u) \in L^1$ *and* $f(x, u)u \in L^1$.

or

(ii) *there is a subsequence of* $\| u_n \|_m$, *which we denote also by* $\| u_n \|_m$, *going to* $+\infty$. *In this case, for a further subsequence,* $v_n = u_n/\| u_n \|_m$ *converges in* H_0^m *to a nonzero* λ_1*-eigenfunction* v *of* L, *and the following inequality holds*

$$\int_\Omega [f(x, u_n) - \lambda_1 u_n] v_n > 0 \tag{19}$$

for n *sufficiently large.*

Proof. As in the previous section we may assume $\lambda_1 > 0$. The existence of solutions u_n of the approximating equations is a direct consequence of Theorem 1. Suppose now that $\| u_n \|_m$ is bounded. Then, going to subsequences if necessary, we conclude that there is $u \in H_0^m$ such that $u_n \to u$, weakly in H_0^m, strongly in L^2, and a.e. in Ω. So $a[u_n , \phi] + 1/n(u_n , \phi)_0 \to a[u, \phi]$ for all $\phi \in C_0^\infty(\Omega)$. In order to prove that u is a distribution solution of $Lu = f(x, u)$

we will show that $f(x, u) \in L^1$ and that $f(x, u_n) \to f(x, u)$ in L^1. Since $f(x, u_n(x)) \to f(x, u(x))$ a.e. in Ω, it suffices to prove that the functions $f(x, u_n(x))$ are equi-absolutely integrable and then use Vitali's theorem. The procedure to be used here is completely analogous to the one in the previous section, and so we omit it. The proof that $f(x, u)u \in L^1$ follows the same lines as the proof of Lemma 4. Finally the proof of the statements on part (ii) follows in a similar way that of Theorem 2 of [2], and again we omit it. Q.E.D.

Now we proceed to give some applications of Theorem 2.

THEOREM 3. *Assume* (L-1), (L-2), (L-3), (f-1), (f-2), *and* (f-4). *Suppose also that for all nonzero eigenfunctions* v *of* L *corresponding to the eigenvalue* λ_1, *we have*

$$\int_{v>0} k_+ v^2 + \int_{v<0} k_- v^2 < \lambda_1 \| v \|_0^2, \tag{20}$$

where k_+ *and* k_- *are defined in* (3). *Then there exists a distribution solution* $u \in H_0{}^m$ *of* $Lu = f(x, u)$, *with* $f(x, u) \in L^1$ *and* $f(x, u)u \in L^1$. (Note that the integrals on the left side of (20) are well defined in $[-\infty, +\infty[$.)

Proof. If alternative (i) of Theorem 2 holds then the present theorem is proved. Alternatively if (ii) of Theorem 2 holds, we proceed as follows to get a contradiction. Inequality (19) gives

$$\lambda_1 \| v_n \|_0^2 < \int_\Omega f(x, u_n) \| u_n \|_m^{-1} v_n$$

for large n. So

$$\lambda_1 \| v \|_0^2 \leqslant \limsup \int_\Omega f(x, u_n) \| u_n \|_m^{-1} v_n \tag{21}$$

and let us estimate the right side of (21). Introduce the following partition of Ω:

$$\begin{aligned} R_n &= \{x \in \Omega : | u_n(x)| \leqslant s_0\} \\ S_n &= \{x \in \Omega : v(x) > 0, | u_n(x)| > s_0\} \\ Z_n &= \{x \in \Omega : v(x) < 0, | u_n(x)| > s_0\} \\ W_n &= \{x \in \Omega : v(x) = 0, | u_n(x)| > s_0\}. \end{aligned}$$

Using (f-2) we have

$$\left| \int_{R_n} f(x, u_n) \| u_n \|_m^{-1} v_n \right| \leqslant \| \alpha_{s_0} \|_0 \| u_n \|_m^{-1} \| v_n \|_0 \to 0.$$

To estimate the integral over S_n, let us denote by χ_n the characteristic function of the set S_n. Observe that, since $u_n(x) = v_n(x) \| u_n \|_m$ and $v_n(x) \to v(x)$, then $u_n(x) \to +\infty$ for $x \in \Omega$ such that $v(x) > 0$. So $\chi_n(x) \to 1$ and $\limsup f(x, u_n(x))\, u_n(x)^{-1} = k_+(x)$, for all x in the set $\{v > 0\}$. Applying these observations and Fatou's lemma to the identity

$$\int_{S_n} f(x, u_n) \| u_n \|_m^{-1} v_n = \int_{v>0} \chi_n \left\{ \frac{f(x, u_n)}{u_n} - (\lambda_1 + \epsilon) - \frac{\beta(x)}{| u_n |} \right\} v_n^2$$
$$+ \int_{v>0} \left(\lambda_1 + \epsilon + \frac{\beta(x)}{| u_n |} \right) \chi_n v_n^2,$$

we get

$$\limsup \int_{S_n} f(x, u_n) \| u_n \|_m^{-1} v_n \leqslant \int_{v>0} k_+ v .$$

In a similar way we prove

$$\limsup \int_{Z_n} f(x, u_n) \| u_n \|_m^{-1} v_n \leqslant \int_{v<0} k_- v^2.$$

The integral over W_n is estimated as follows:

$$\int_{W_n} f(x, u_n) \| u_n \|_m^{-1} v_n \leqslant \int_{W_n} \left(\lambda_1 + \epsilon + \frac{\beta(x)}{| u_n |} \right) v_n^2$$
$$\leqslant (\lambda_1 + \epsilon) \int_{v=0} v_n^2 + \| u_n \|_m^{-1} \int_\Omega \beta \, | v_n | \to 0.$$

Consequently (21) gives

$$\lambda_1 \| v \|_0^2 \leqslant \int_{v>0} k_+ v^2 + \int_{v<0} k_- v^2,$$

yielding in this manner a contradiction to our assumption (20). Q.E.D.

COROLLARY 1. *Assume* (L-1), (L-2), (L-3), (f-1), (f-2), (f-4), *and that the solutions of* $Lu = \lambda_1 u$ *have the unique continuation property. That is, if* $u \in H_0^m$ *is a distribution solution of* $Lu = \lambda_1 u$, *which vanishes on a subset of* Ω *with positive measure, then* $u \equiv 0$. *Suppose that there exists a subset* Ω' *of* Ω *with positive measure, such that*

$$k_+(x), k_-(x) < \lambda_1 , \qquad x \in \Omega'.$$

Then there exists a distribution solution $u \in H_0^m$ *of* $Lu = f(x, u)$, *with* $f(x, u) \in L^1$ *and* $f(x, u)u \in L^1$.

Proof. It suffices to show that under the above hypotheses condition (20) of Theorem 3 holds true. Suppose by contradiction that this is not the case, i.e.,

$$\int_{v>0} (k_+ - \lambda_1)\, v^2 + \int_{v<0} (k_- - \lambda_1)\, v^2 \geqslant 0,$$

for some nonzero λ_1-eigenfunction v. Using the fact that (f-4) implies k_+, $k_- \leqslant \lambda_1$, in Ω, we conclude that $k_+ = \lambda_1$ in $v > 0$ and $k_- = \lambda_1$ in $v < 0$. This together with the fact that the set $\{v = 0\}$ has measure zero, gives a contradiction. Q.E.D.

COROLLARY 2. *Assume* (L-1), (L-2), (L-3), (f-1), (f-2), (f-4), *and that there exists a subset* Ω' *of* Ω, *with positive measure, where* $k_+ < \lambda_1$. *Suppose also that* (20) *holds* (*only*) *for the nonzero* λ_1-*eigenfunctions* v *which are* $\leqslant 0$ *on* Ω'. *Then there exists a distribution solution* $u \in H_0^m$ *of* $Lu = f(x, u)$, *with* $f(x, u)$ *and* $f(x, u)u$ *in* L^1.

Proof. It suffices to show that under the present hypotheses condition (20) of Theorem 3 holds true. Proceeding as in the proof of the previous corollary, we conclude that $k_+ = \lambda_1$ on $v > 0$ and $k_- = \lambda_1$ on $v < 0$. In view of our hypotheses it follows that $v(x) \leqslant 0$ on Ω', and this leads to a contradiction with our assumption. Q.E.D.

Of course an analogous result holds assuming the following hypothesis on k_-, instead of k_+: $k_- < \lambda_1$ on some subset Ω' of Ω, with positive measure.

Some Examples of Application of Theorem 3 *and Its Corollaries*

(1) Suppose $k_+ < \lambda_1$ on the whole of Ω; as a special case one could have $k_+ \equiv -\infty$. Then, by Corollary 2, a solution exists provided

$$\int_\Omega k_- v^2 < \lambda_1 \| v \|_0^2 \tag{22}$$

for all nonzero λ_1-eigenfunctions $v \leqslant 0$.

(2) In many applications the function $f(x, u)$ appears in the form $k(x, u)u + h(x)$, where $h(x) \in L^2(\Omega)$ and k satisfies Carathéodory's conditions and a hypothesis like (f-2). So a nontrivial example of application of Corollary 1, when $\lambda_1 = 0$, would be in the case that $k(x, u) = -(e^u + 1)\chi_{\Omega'}$, where $\chi_{\Omega'}$ is the characteristic function of a subset Ω' of Ω with $0 < |\Omega'| < |\Omega|$. Indeed, $k_+(x) = -\infty$ on Ω', $k_+(x) = 0$ on $\Omega \backslash \Omega'$, $k_-(x) = -1$ on Ω', and $k_-(x) = 0$ on $\Omega \backslash \Omega'$.

Our next application is to equations of the form

$$Lu - \lambda_1 u = g(x, u) + h(x), \tag{23}$$

where $h \in L^2(\Omega)$ and g satisfies the following hypotheses:

(g-1) Carathéodory's conditions.

(g-2) For each $r > 0$ there exists $\alpha_r(x) \in L^2(\Omega)$ such that

$$|g(x, s)| \leqslant \alpha_r(x), \qquad x \in \Omega, \quad |s| \leqslant r.$$

(g-3) $\limsup_{|s|\to\infty} g(x, s)/s \leqslant 0$, uniformly in $x \in \Omega$.

(g-4) There exist two L^2-functions $a(x) \geqslant 0$ and $b(x) \leqslant 0$ such that

$$g(x, s) \leqslant a(x), \qquad x \in \Omega, \quad s \geqslant 0 \tag{24}$$

$$g(x, s) \geqslant b(x), \qquad x \in \Omega, \quad s \leqslant 0. \tag{25}$$

Remark. Condition (g-4) is a stronger restriction on the one-sided growth of g than (g-3) is. Indeed, compare (24) and (25) with the following equivalent statement of (g-3): given $\epsilon > 0$ there is $s_0 > 0$ and $0 \leqslant \beta(x) \in L^2(\Omega)$ such that

$$g(x, s) \leqslant \epsilon s + \beta(x), \qquad x \in \Omega, \quad s \geqslant s_0$$
$$g(x, s) \geqslant \epsilon s - \beta(x), \qquad x \in \Omega, \quad s \leqslant -s_0 .$$

Observe however that g is unrestricted from below as $s \to +\infty$. And similarly g is unrestricted from above as $s \to -\infty$. Finally note that (g-2) implies that condition (g-4) is equivalent to the requirement that (24) and (25) hold only for $|s| \geqslant$ some s_0.

THEOREM 4. *Assume* (L-1), (L-2), (L-3), (g-1), (g-2), *and* (g-4). *Then, for a given* $h \in L^2$, *there exists a distribution solution* $u \in H_0^m$ *of* (23) *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1, *provided*

$$\int_\Omega h(x)\, v(x) < -\int_{v>0} G_+(x)\, v(x) - \int_{v<0} G_-(x)\, v(x) \tag{26}$$

for all nonzero λ_1*-eigenfunctions* v *of* L, *where*

$$G_+(x) = \limsup_{s\to+\infty} g(x, s)$$

and

$$G_-(x) = \liminf_{s\to-\infty} g(x, s).$$

Remarks. (a) Note that the integrals on the right side of (26) are well defined, with values in $[-\infty, \infty[$. Note also that if $G_-(x) \geqslant g(x, s) \geqslant G_+(x)$, for all $x \in \Omega$, $s \in \mathbb{R}$, then condition (26) with $<$ replaced by $\leqslant$ is essentially necessary for the existence of a solution $u \in H_0^m$ of (23) with $g(x, u)$ in L^1. Indeed, assuming some regularity on L and Ω (or that $2m > n$, and Ω satisfies the cone condition) so that any λ_1-eigenfunction v of L is continuous on $\bar{\Omega}$, one deduces from (23) that

$$\int_\Omega h(x)\, v(x)\, dx = -\int_\Omega g(x, u)\, v(x)\, dx$$

and the conclusion follows readily.

(b) Theorems 3 and 4 are not deductible one from the other. We next give an example where Theorem 3 applies but Theorem 4 does not. Let the function f in (1) be given by:

$$\begin{aligned} f(x, s) &= c_+(x)\, s + d_+(x)\, g_+(s), \qquad s \geqslant 0 \\ &= c_-(x)\, s + d_-(x)\, g_-(s), \qquad s \leqslant 0 \end{aligned}$$

where $0 \geqslant c_\pm \in L^2(\Omega)$, $d_\pm \in L^\infty(\Omega)$, $g_\pm$ are continuous with $g_\pm(0) = 0$, $g_+(s)s^{-1} \to 0$ as $s \to +\infty$ and $g_-(s)s^{-1} \to 0$ as $s \to -\infty$. Assume also that L in (1) is a second-order uniformly strongly elliptic operator with $\lambda_1 = 0$; this implies that the λ_1-eigenfunctions are of the same sign in Ω. Then, assuming that $c_\pm \neq 0$ on sets of positive measure, we see that Theorem 3 applies to guarantee that there exists a distribution solution $u \in H_0{}^m$ of $Lu = f(x, u)$. We also see that Theorem 4 does not apply since condition (g-4) is not verified in this case. On the other hand, the next example exhibits a situation where Theorem 4 applies but Theorem 3 does not. Let the function f in (1) be given by

$$\begin{aligned} f(x, s) &= \quad a_+(x) + b_+(x)\, g_+(s), \qquad s \geqslant 0 \\ &= -a_-(x) - b_-(x)\, g_-(s), \qquad s \leqslant 0 \end{aligned}$$

where $0 \geqslant a_\pm \in L^2(\Omega)$, $b_\pm \in L^2(\Omega)$, $g_\pm$ are continuous, with $g_\pm(0) = 0$, $g_\pm(s) \to 0$ as $s \to \pm\infty$. Assume also that L in (1) is a second-order uniformly strongly elliptic operator with $\lambda_1 = 0$. Then, assuming that $a_\pm(x) \neq 0$ on sets of positive measure, we see that (26) holds, and consequently Theorem 4 applies. We also see that $k_\pm(x) \equiv 0$ and then Theorem 3 does not apply.

Proof. Let $f(x, u) = \lambda_1 u + g(x, u) + h(x)$. Such an f satisfies all the hypotheses of Theorem 2. So if alternative (i) of Theorem 2 holds, then (23) has a distribution solution $u \in H_0{}^m$ with $g(x, u)$ and $g(x, u)u$ are in $L^1(\Omega)$. To complete the proof let us show that (26) implies that alternative (ii) of Theorem 2 cannot hold. Accordingly suppose by contradiction that (19) holds, which implies

$$\int_\Omega h(x)\, v_n > \int_\Omega -g(x, u_n)\, v_n\,. \tag{27}$$

The left side of (27) converges to $\int_\Omega h(x)v$; to evaluate the lim inf of the right side of (27) let us break the integral over Ω as a sum of six integrals whose lim inf's will be calculated separately.

(i) $$\int_{v>0,\, u_n \geqslant 0} = \int_{v>0} \chi_n[-g(x, u_n)\, v_n + a(x)\, v_n] - \int_{v>0} \chi_n a(x)\, v_n$$

where χ_n is the characteristic function of the set $\{u_n \geqslant 0\}$. Since $u_n =$

$v_n \| u_n \|_m \to +\infty$ in $\{v > 0\}$, it follows that $\chi_n \to 1$ a.e. in $\{v > 0\}$, and by Fatou's lemma

$$\begin{aligned}
\liminf \int_{v>0} & \chi_n[-g(x, u_n)\, v_n + a(x)\, v_n] \\
&\geqslant \int_{v>0} \liminf \chi_n[-g(x, u_n)\, v_n + a(x)\, v_n] \\
&\geqslant -\int_{v>0} \limsup \chi_n g(x, u_n)\, v_n + \int_{v>0} \liminf \chi_n a(x)\, v_n \\
&= -\int_{v>0} G_+(x)\, v(x) + \int_{v>0} a(x)\, v.
\end{aligned}$$

Next we observe that

$$\lim \int_{v>0} \chi_n a(x)\, v_n = \int_{v>0} a(x)\, v$$

and consequently

$$\liminf \int_{v>0,\, u_n \geqslant 0} \geqslant -\int_{v>0} G_+(x)\, v(x).$$

(ii) $$\int_{v<0,\, u_n<0} = \int_{v<0} \varphi_n[-g(x, u_n)\, v_n + b(x)\, v_n] - \int_{v<0} \varphi_n b(x)\, v_n\,,$$

where φ_n is the characteristic function of $\{u_n < 0\}$. Since $u_n = v_n \| u_n \|_m \to -\infty$ in $\{v < 0\}$, it follows that $\varphi_n \to 1$ a.e. in $\{v < 0\}$. By arguments similar to the ones in (i) above we conclude that

$$\liminf \int_{v<0,\, u_n<0} \geqslant -\int_{v<0} G_-(x)\, v(x).$$

(iii) $$\int_{v>0,\, u_n<0} = \int_{v>0} -\varphi_n g(x, u_n)\, v_n \geqslant \int_{v>0} -\varphi_n b(x)\, v_n\,,$$

where the last integral converges to zero in view of the observation that $\varphi_n \to 0$ a.e. in $\{v > 0\}$. So

$$\liminf \int_{v>0,\, u_n<0} \geqslant 0.$$

(iv) $$\int_{v<0,\, u_n \geqslant 0} = \int_{v<0} -\chi_n g(x, u_n)\, v_n \geqslant \int_{v<0} -\chi_n a(x)\, v_n\,,$$

and by an argument similar to the one in (iii) we get

$$\liminf \int_{v<0,\, u_n \geqslant 0} \geqslant 0.$$

(v) $$\int_{v=0,\, u_n \geqslant 0} = \int_{v=0} -\chi_n g(x, u_n)\, v_n \geqslant \int_{v=0} -\chi_n a(x)\, v_n\,,$$

where the last integral converges to zero since $v_n \to v$ in $L^2(\Omega)$ and thus $v_n \to 0$ in $L^2(\{v = 0\})$. So

$$\liminf \int_{v=0,\, u_n \geqslant 0} \geqslant 0.$$

$$\text{(vi)} \quad \int_{v=0,\, u_n<0} = \int_{v=0} -\varphi_n g(x, u_n)\, v_n \geqslant \int_{v=0} -\varphi_n b(x)\, v_n$$

and we proceed as in (v). So

$$\liminf \int_{v=0,\, u_n<0} \geqslant 0.$$

Putting together the estimates in (i),..., (vi) above, we get from (27)

$$\int_\Omega h(x)\, v(x) \geqslant -\int_{v>0} G_+(x)\, v(x) - \int_{v<0} G_-(x)\, v(x),$$

which contradicts (26). The proof of Theorem 4 is complete. Q.E.D.

Corollary 3. *Assume* (L-1),..., (g-4) *as in Theorem* 4. *Then, for any given* $h \in L^2$ *orthogonal to the* λ_1*-eigenspace of* L, *there exists a distribution solution* $u \in H_0^m$ *of* (23), *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1, *provided*

$$G_-(x) > 0 > G_+(x), \qquad x \in \Omega.$$

Proof. If (26) does not hold, then

$$0 \geqslant -\int_{v>0} G_+ v - \int_{v<0} G_- v$$

for some nonzero λ_1-eigenfunction v of L. But both $-G_+v$ and $-G_-v$ in the above integrals are $\geqslant 0$. Consequently $G_+v = 0$ on $\{v > 0\}$ and $G_-v = 0$ on $\{v < 0\}$, and thus $v = 0$ a.e. in Ω, which is a contradiction. Q.E.D.

Corollary 4. *Assume* (L-1),..., (g-4) *as in Theorem* 4. *Suppose that* $G_+(x) = -\infty$ *on a subset* Ω' *of* Ω, *with positive measure. Then, for a given* $h \in L^2$, (23) *has a distribution solution* $u \in H_0^m$ *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1 *provided*

$$\int_\Omega hv < -\int_{v>0} G_+ v - \int_{v<0} G_- v \tag{28}$$

for all nonzero λ_1*-eigenfunctions* $v \leqslant 0$ *in* Ω'.

Proof. We claim that (26) holds (for *all* nonzero λ_1-eigenfunctions). Suppose by contradiction that there is $v \neq 0$ such that

$$\int_\Omega hv \geqslant -\int_{v>0} G_+ v - \int_{v<0} G_- v. \tag{29}$$

Since $G_+ = -\infty$ in Ω', (29) implies that $v \leqslant 0$ in Ω'. Thus (29) would hold for a nonzero λ_1-eigenfunction $v \leqslant 0$ in Ω', which contradicts (28). Q.E.D.

COROLLARY 5. *Assume* (L-1),..., (g-4) *as in Theorem* 4. *Suppose that* $G_+ = -\infty$ *in the whole of* Ω. *Then, for a given* $h \in L^2$, *a distribution solution* $u \in H_0{}^m$ *of* (23), *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1, *exists provided*

$$\int_\Omega hv < -\int_\Omega G_- v \tag{30}$$

for all nonzero λ_1*-eigenfunction* $v \leqslant 0$ *in* Ω.

Proof. Special case of Corollary 3. Q.E.D.

Of course results analogous to Corollaries 4 and 5 hold for $G_-(x) = +\infty$ (instead of $G_+(x) = -\infty$) on some $\Omega' \subset \Omega$.

COROLLARY 6. *Assume* (L-1),..., (g-4) *as in Theorem* 4. *Suppose also that* $G_+ = -\infty$ *and* $G_- = +\infty$ *in the whole of* Ω. *Then a distribution solution* $u \in H_0{}^m$ *of* (23), *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1, *exists for all* $h \in L^2$.

Proof. Direct consequence of Corollary 5. Q.E.D.

The conclusion of Corollary 6 still holds if $G_+ = -\infty$ and $G_- = +\infty$ on $\Omega' \subset \Omega$ with $|\Omega'| > 0$, provided that the solutions of $Lu = \lambda_1 u$ enjoy the unique continuation property. This follows from (26).

COROLLARY 7 (Landesman–Lazer [9]). *Suppose* (L-1), (L-2), (L-3), *and* (g-1). *Assume also that there is* $c(x) \in L^2(\Omega)$ *such that* $|g(x, s)| \leqslant c(x)$, *for all* $s \in \mathbb{R}$, $x \in \Omega$. *Then, for a given* $h \in L^2$, *Eq.* (23) *has a distribution solution* $u \in H_0{}^m$, *with* $g(x, u)$ *and* $g(x, u)u$ *in* L^1, *provided*

$$\int_\Omega hv < -\int_{v>0} G_+ v - \int_{v<0} G_- v \tag{31}$$

for all nonzero λ_1*-eigenfunctions.*

Proof. It suffices to observe that, under the present hypotheses, conditions (g-2), (g-3), and (g-4) follow immediately. Q.E.D.

COROLLARY 8. (*A nonresonant problem, a reformulation of Theorem* 1). *Assume* (L-1), (L-2), (L-3), (g-1), *and* (g-2). *Suppose also that there are* $\epsilon > 0$ *and* $s_0 > 0$ *and* $0 \leqslant \beta(x) \in L^2(\Omega)$ *such that*

$$g(x, s)/s \leqslant -\epsilon + \beta(x)/|s|, \qquad x \in \Omega, \quad |s| > s_0. \tag{32}$$

Then (23) *has a distribution solution* $u \in H_0^m$, *with* $g(x, u)$ *and* $g(x, u)u$ *in* $L^1(\Omega)$, *for each* $h \in L^2(\Omega)$ *given.*

Proof. (32) implies (g-4). Moreover $G_+ \equiv -\infty$, $G_- \equiv +\infty$. So the result follows from Corollary 6. Q.E.D.

Some Examples of Application of Theorem 4 *and Its Corollaries*

(I) Let $g(x, s) = (-s)^{1/2}$, for $s \leqslant 0$ and $g(x, s) = -se^s$, for $s > 0$. Then for each given $h \in L^2$, Eq. (23) has a distribution solution $u \in H_0^m$, with $g(x, u)$ and $g(x, u)u$ in $L^1(\Omega)$. To see that, just observe that $G_+ \equiv -\infty$ and $G_- \equiv +\infty$. All the other conditions of Corollary 6 are readily verified. Let us remark that this is a resonant problem since $\lim_{s\to-\infty} g(x, s)/s = 0$.

(II) Let $g(x, s) = -g(x)e^s$, where $g(x) \in L^2(\Omega)$, $g(x) \geqslant 0$, and the set $\Omega' = \{x \in \Omega : g(x) > 0\}$ has positive measure. Then, for a given $h \in L^2(\Omega)$, there exists a distribution solution $u \in H_0^m$ of $Lu - \lambda_1 u = -g(x)e^u + h(x)$, with $g(x, u)$ and $g(x, u)u$ in L^1, provided $\int_\Omega hv < 0$ for all eigenfunctions $v \leqslant 0$ in Ω'. To prove this we use Corollary 4. Since $G_+ = -\infty$ in Ω', $G_+ = 0$ in $\Omega\backslash\Omega'$, and $G_- \equiv 0$, (30) reduces to $\int_\Omega hv < 0$. Again observe that this is a resonant problem since $\lim_{s\to-\infty} g(x, s)/s = 0$, for $x \in \Omega$, and $\lim_{s\to+\infty} g(x, s)/s = -\infty$, for $x \in \Omega'$. This example has been studied in [3] for second-order operators.

(III) Let $g(x, s) = g(x)e^{-s}$, where $g(x) \in L^2$, $g(x) \geqslant 0$, and let $\Omega' = \{x \in \Omega : g(x) > 0\}$. Then, for a given $h \in L^2(\Omega)$, there exists a distribution solution $u \in H_0^m$ of $Lu - \lambda_1 u = g(x)e^{-u} + h(x)$, with ge^{-u} and $ge^{-u}u$ in L^1, provided $\int_\Omega hv < 0$ for all nonzero λ_1-eigenfunction $v \geqslant 0$ in Ω'. Indeed, $G_+(x) \equiv 0$, $G_-(x) = +\infty$ on Ω' and 0 on $\Omega\backslash\Omega'$, and it suffices to apply Corollary 4 (with the roles of G_+ and G_- interchanged). Again this is a resonant problem.

REFERENCES

1. D. G. DE FIGUEIREDO, "The Dirichlet Problem for Nonlinear Elliptic Equations: A Hilbert Space Approach," Springer Lecture Notes, Vol. 445, pp. 144–165, Springer-Verlag, New York, 1974.
2. D. G. DE FIGUEIREDO, Semilinear elliptic problems with nonlinearities near the first eigenvalue, *in* "Dynamical Systems" (Proc. Univ. Florida International Symposium) (A. R. Bednarek and L. Cesari, Eds.), pp. 59–70, Academic Press, New York, 1976.
3. J. KAZDAN AND F. W. WARNER, Remarks on some quasilinear elliptic equations, *Comm. Pure Appl. Math.* **28** (1975), 567–597.
4. J. P. GOSSEZ, Nonlinear elliptic boundary value problems for equations with rapidly or slowly increasing coefficients, *Trans. Amer. Math. Soc.* **190** (1974), 163–205.

5. J. P. GOSSEZ, Surjectivity results for pseudo-monotone mappings in complementary systems, *J. Math. Anal. Appl.* **53** (1976), 484–494.
6. D. G. DE FIGUEIREDO AND J. P. GOSSEZ, Perturbation non linéaire d'un problème elliptique linéaire près de sa première valuer propre, *C. R. Acad. Sci. Paris* **284** (1977), 163–166.
7. A. FRIEDMAN, "Partial Differential Equations," Holt, Rinehart & Winston (1969).
8. C. L. DOLPH, Nonlinear integral equations of the Hammerstein type, *Trans. Amer. Math. Soc.* **66** (1949), 289–307.
9. E. M. LANDESMAN AND A. C. LAZER, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* **19** (1970), 609–623.
10. L. NIRENBERG, An application of generalized degree to a class of nonlinear problems, *Troisième Colloq. Anal. Fonct. Liege* (Sept. 1970), 57–74.
11. PETER HESS, On a theorem by Landesman and Lazer, *Indiana Univ. Math. J.* **23** (1974), 827–829.
12. S. FUCIK, Open problems in the solvability of nonlinear equations, to appear.
13. A. AMBROSETTI AND G. PRODI, On the inversion of some differentiable mappings with singularities between Banach spaces, *Ann. Mat.* **93** (1972).
14. D. SCHWARTZ, "Linear Operators," Part 1: "General Theory," Interscience, New York, 1958.
15. W. A. STRAUSS, On weak solutions of semi-linear hyperbolic equations, *An. Acad. Brasil Ci.* **42** (1970), 645–651.
16. H. BREZIS AND L. NIRENBERG, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, *Ann. Scuola Norm. Sup. Pisa*, to appear.